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INVESTIGATION OF STABILITY OF THE UNPERTURBED MOTION OF AN AXISYMMETRIC SOLID WHEN ITS CENTER OF MASS MOVES SPATIALLY IN THE AIR

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Free motion of an axisymmetric solid is considered when its center of mass undergoes a spatial displacement along a curve of double curvature, and a rotational motion about this center of mass is also taken into account. The equations of motion are constructed using the tangential and normal components of the drag, the Magnus force and the weight of the solid as well as the tilting (restoring) moment, Magnus force moment, axial and equatorial damping moments. Conditions are established under which the deviations of the symmetry axis of the body from the tangent to the trajectory of its center of mass will not exceed some specified values over the given interval of time.

The freely moving solid has the rigidly attached $C\xi\eta\zeta$ -axes which represent the principal central axes of its inertia ellipsoid. The ellipsoid is a solid of revolution, and $C\xi$ is its axis of symmetry. The center of mass moves along a curve of double curvature; its velocity vector forms an angle γ with the vertical CXY plane, and an angle Θ with the horizontal plane (see Fig. 1). The figure also depicts the velocity semiaxes $CX'Y'Z'$, the velocity axes $CX'Y_2Z_2$ [1], the intermediate axes $C\xi\eta'\zeta'$, the Euler angles δ, φ, ψ and the angles $\delta_1, \delta_2, \delta_3$ which are used to define the position of the $C\xi\eta\zeta$ axes relative to $CX'Y'Z'$.

Using the accepted assumptions (see e.g. [1, 2]), we apply the following forces to the body: the weight Q, the tangential R_T and normal R_N components of the drag, and the Magnus force R_L, and we have

$$R_N = \left(\frac{\partial R_N}{\partial \delta}\right)_0 \delta + \frac{1}{3!}\left(\frac{\partial^3 R_N}{\partial \delta^3}\right)_0 \delta^3 + \ldots, \quad R_L = c_L p v \sin \delta$$

In addition we apply to the body the tilting moment $(B\beta \sin \delta)$ if the center of pressure is above the center of mass, or the stabilizing moment $(-B\beta\sin\delta)$ otherwise, the Magnus force moment $M_L = \pm h_L R_L$ (with the sign chosen similarly), and damping moment $(-A\chi p, -2B\varkappa q, -2B\varkappa r)$. Here A and B denote the axial and equatorial moments of inertia of the body, v is the velocity of the center of mass $\chi, \varkappa, \beta$ and c_L are variable proportionality coefficients, h_L is the shoulder of the Magnus force, p is the angular velocity of rotation of the body about its symmetry axis, and q, r are the projections of the angular velocity of the symmetry axis of the body on the $C\eta$ and $C\zeta$ axes.

The equations of motion of the center of mass in terms of the projections on the $CX'Y'Z'$ axes are

$$mv^\cdot = -R_T - Q\sin\Theta, \quad mv\Theta^\cdot = R_N\cos\psi + R_L\sin\psi - Q\cos\Theta, \quad mv\gamma^\cdot\cos\Theta = R_N\sin\psi - R_L\cos\psi$$

Taking into account the fact that $\delta \cos\psi = \delta_1 + \ldots, \quad \delta \sin\psi = \delta_2 + \ldots$ (obtained from the spherical triangle $\xi X' X_*$), we have

$$\Theta^{\cdot} = 2\mu\delta_1 + 2\xi\delta_2 - gv^{-1}\cos\Theta + \ldots, \quad \gamma^{\cdot}\cos\Theta = 2\mu\delta_2 - 2\xi\delta_1 + \ldots \qquad (1)$$
$$2\mu = (\partial R_N / \partial\delta)_0 (mv)^{-1}, \quad 2\xi = c_L p m^{-1}$$

where the repeated dots denote the third and higher order of smallness terms in δ_1 and δ_2.

The equations of rotational motion are the dynamic Euler equations

$$Ap^{\cdot} = -A\chi p, \quad Bq^{\cdot} + (B - A)rp = \pm M_L \cos\varphi - 2B\varkappa q \pm B\beta \sin\delta \sin\varphi$$

$$Br^{\cdot} + (A - B)pq = \mp M_L \sin\varphi - 2B\varkappa r \pm B\beta \sin\delta \cos\varphi$$

$$p = (\delta_1^{\cdot} + \Theta^{\cdot})\sin\delta_2 + \delta_3^{\cdot} - \gamma^{\cdot}\sin(\delta_1 + \Theta)\cos\delta_2, \quad q = (\delta_1^{\cdot} + \Theta^{\cdot})\sin\delta_3\cos\delta_2 - \delta_2^{\cdot}\cos\delta_3 + \gamma^{\cdot}\cos(\delta_1 + \Theta)\cos\delta_3 + \gamma^{\cdot}\sin(\delta_1 + \Theta)\sin\delta_2\sin\delta_3, \quad r = (\delta_1^{\cdot} + \Theta^{\cdot})\cos\delta_3\cos\delta_2 + \delta_2^{\cdot}\sin\delta_3 + \gamma^{\cdot}\cos(\delta_1 + \Theta)\sin\delta_3 + \gamma^{\cdot}\sin(\delta_1 + \Theta)\sin\delta_2\cos\delta_3$$

Integrating the first equation we obtain

$$p = p_0 \exp\left(-\int_0^t \chi dt\right)$$

while the second and third equations become, after substituting p, q and r, performing

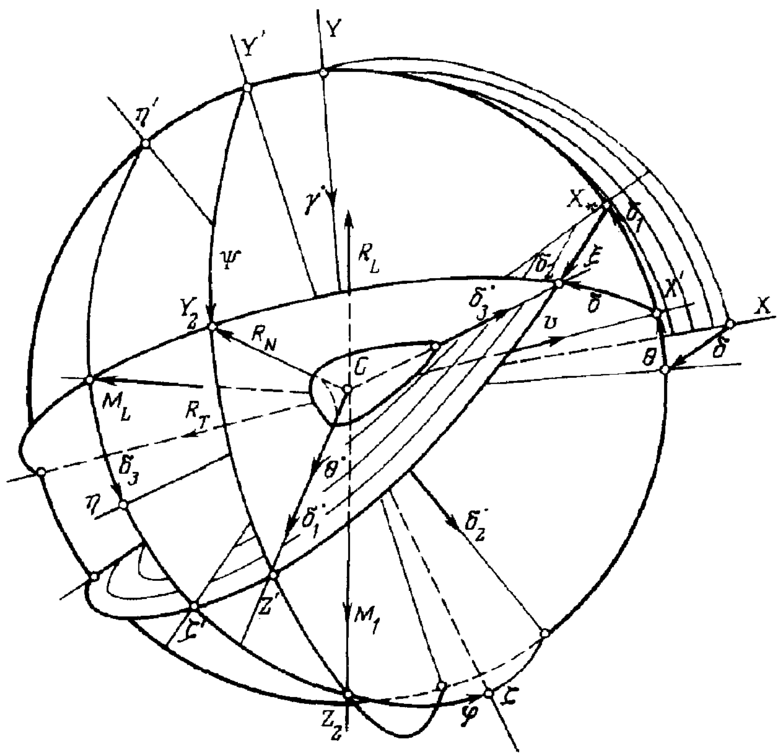

Fig. 1

the necessary manipulations and expanding into trigonometric series near $\delta_1 = \delta_2 = 0$, with (1) and equations obtained by differentiating (1) with respect to time all taken into account,

$$\begin{aligned} &\delta_1^{\cdot\cdot} + 2a\delta_2^{\cdot} + 2b\delta_1^{\cdot} - c\delta_1 - e\delta_2 = R_1 + \Psi_1 \\ &\delta_2^{\cdot\cdot} - 2a\delta_1^{\cdot} + 2b\delta_2^{\cdot} - c\delta_2 + e\delta_1 = R_2 + \Psi_2 \end{aligned} \qquad (2)$$

$$a = \alpha + \xi,\ b = \varkappa + \mu,\ c = \pm\beta - 2\mu^{\cdot} - 4\chi\mu - 4\alpha\xi,\ e = \pm\nu - 2\xi^{\cdot} - 4\chi\xi,\ -4\alpha\mu$$

$$\alpha = \frac{1}{2}\frac{A}{B}\ p,\ \nu = h_L c_L p v B^{-1},\ R_1 = 2\varkappa g v^{-1}\cos\Theta - g\,(v^{-1}\cos\Theta),\ R_2 = -2\alpha g v^{-1}\cos\Theta$$

where Ψ and Ψ_2 denote the nonlinear terms in the expansions.

Let us consider, together with (2), a reduced system of equations

$$\delta_1^{\cdot\cdot} + 2a\delta_2^{\cdot} + 2b\delta_1^{\cdot} - c\delta_1 - e\delta_2 = 0, \quad \delta_2^{\cdot\cdot} - 2a\delta_1^{\cdot} + 2b\delta_2^{\cdot} - c\delta_2 + e\delta_1 = 0 \qquad (3)$$

which admits the particular solution

$$\delta_1 = \delta_2 = 0,\ \delta_1^{\cdot} = \delta_2^{\cdot} = 0$$

corresponding to a helical motion of the symmetry axis of the body along the tangent to the trajectory of the center of mass. The system (2) differs from (3) in the appearance of the nonlinear terms Ψ_1 and Ψ_2 and of continuously acting perturbations R_1 and R_2 caused by lowering of the tangent.

Let us determine the conditions of stability of the unperturbed motion (4) both in the presence and absence of the continuously acting perturbations R_1 and R_2 and nonlinear terms Ψ_1 and Ψ_2. When the coefficients are constant, the characteristic equation of the system (3) is

$$\lambda^4 + 4b\lambda^3 + (4b^2 - 2c + 4a^2)\lambda^2 - 4\,(bc + ea)\lambda + c^2 + e^2 = 0$$

Applying the Hurwitz criterion, we obtain the following conditions for the asymptotic stability of the solution (4):

a) If $c > 0$ (body without fins), then

$$a^2 - c > 0, \quad e < 0, \quad 2a\,(1-\sigma) \leqslant \left|\frac{e}{b}\right| \leqslant 2a\,(1+\sigma), \quad \sigma = \sqrt{1 - \frac{c}{a^2}} < 1$$

b) If $c < 0$ (finned body), then

$$e \gtrless 0, \quad -2a\,(\sigma+1) \leqslant \frac{e}{b} \leqslant 2a\,(\sigma - 1), \quad |c| > \frac{ae}{b} + \frac{1}{4}\left(\frac{e}{b}\right)^2, \quad \sigma > 1$$

Therefore in the case a) the free solid must have a considerable angular velocity about its symmetry axis, while in the case b) the modulus of the coefficient e should be large.

The conclusions remain valid when the nonlinear terms Ψ_1 and Ψ_2 are taken into account [3]. If on the other hand we take into account the continuously acting perturbations R_1 and R_2, the unperturbed motion (4) will be simply stable [3].

When the coefficients are all variable, we introduce the function

$$V(t, \delta_1, \delta_2, \delta_1^{\cdot}, \delta_2^{\cdot}) = \delta_1^{\cdot 2} + \lambda\delta_1^{\cdot}\delta_2 + (\lambda a - c)\delta_2^2 + \delta_2^{\cdot 2} - \lambda\delta_2^{\cdot}\delta_1 + (\lambda a - c)\delta_1$$

where λ is a parameter to be defined. Clearly, $V(t, 0, 0, 0, 0) = 0$ and V is a positive definite function provided that the generalized Silvester conditions [4]

$$\lambda a - c > k_1 > 0, \; a^2 - c > k_2 > 0, \; a \; \lambda \in [2a(1-\sigma), \; 2a(1+\sigma)] \qquad (5)$$

hold. The time derivative

$$V^{\cdot} = -[4b\delta_1^{\cdot 2} + 2(b\lambda - e)\delta_1^{\cdot}\,\delta_2 - (\lambda e + \lambda a^{\cdot} - c^{\cdot})\delta_2^2 + 4b\delta_2^{\cdot 2} - 2(b\lambda - e)\delta_2^{\cdot}\delta_1 - (\lambda e + \lambda a^{\cdot} - c^{\cdot})\delta_1^2]$$

will be, by virtue of Eqs. (3), negative definite if

$$-(\lambda e + \lambda a^{\cdot} - c^{\cdot}) > k_3 > 0, \; a^{\cdot 2} + ea^{\cdot} + bc^{\cdot} > k_4 > 0 \qquad (6)$$

Moreover, a positive definite function V and a negative definite $V^{\cdot}$ can be constructed by virtue of Eqs. (3) with the help of one and the same parameter λ only when

$$e < 0, \quad 2a\left(1 - \sigma - \frac{\chi}{b}\right) < -\frac{e}{b} < 2a\left(1 + \sigma - \frac{\chi}{b}\right)$$

$$e \gtrless 0, \quad -2a(\sigma + 1 - \frac{\chi}{b}) \leqslant \frac{e}{b} < 2a\left(\sigma - 1 - \frac{\chi}{b}\right)$$

where $e < 0$ for a body without fins and $\varepsilon > 0$ for a finned body.

Integrating the first inequality of (6) over the interval $[0, T]$ and taking the first inequality of (5) into account, we have

$$0 < k_1 < \lambda a - c < \lambda a_0 - c_0 - k_3 T - \lambda \int_0^T e dt$$

In this case the partial derivatives $\partial V/\partial\delta_1$, $\partial V/\partial\delta_2$, $\partial V/\partial\delta_1^{\cdot}$, $\partial V/\partial\delta_2^{\cdot}$ are bounded in the interval $[0, T]$ provided that $|\delta_i| < \varepsilon$, $|\delta_i^{\cdot}| < \varepsilon$, $i = 1, 2$ where $e > 0$ is an arbitrarily small preassigned number. Consequently we find, in accordance with the Malkin theorem [3] that for the interval $[0, T]$ the representative point $(\delta_1, \delta_2, \delta_1^{\cdot}, \delta_2^{\cdot})$ occurring within the region $V_0 = V(0, \delta_{10}, \delta_{20}, \delta_{10}^{\cdot}, \delta_{20}^{\cdot})$ at the initial instant of time, will remain there for $t \in [0, T]$ provided that $|R_i| < \zeta(\varepsilon)$, $|\delta_{i0}| < \eta(\varepsilon)$, $|\delta_{i0}^{\cdot}| < \eta(\varepsilon)$ $(i = 1, 2)$.

If the interval $[0, T]$ is chosen in accordance with the second inequalities of (5) and (6), then the solution of (2) will not emerge, in the given interval, from the closed region V_0 and this means that the unperturbed motion (4) is stable in the interval $[0, T]$ in the presence, as well as the absence of continuously acting perturbations and nonlinear terms. In this case the system (2) can be linearized. Introducing the complex variable $W = \delta_1 + i\delta_2$, we can write the linearized system in the form of a single equation

$$W^{\cdot\cdot} - 2(ia - b)W^{\cdot} - (c - ei)W = R_1 + iR_2 \qquad (7)$$

When $c > 0$, Eq. (7) contains a large parameter $a_0 = a_0$, and when $c < 0$ a large parameter $|c_0| = |c(0)|$, consequently its solution can be constructed using the asymptotic method [5]. We have, with the accuracy to within the values of λ

where $\lambda = a_0^{-1}$ in the first case and $\lambda = |c_0|^{-\frac{1}{2}}$ in the second case,

$$W = \frac{1}{\sqrt{\tau}}\left[C_1 \exp \int_0^t (i\lambda\tau + ia - b)\,dt + C_2 \exp \times \int_0^t (-i\lambda\tau + ia - b)\,dt\right] - \frac{R_1 + iR_2}{c - ei}$$

$$\tau^2 = [(a^2 - c - b^2 - b^{\cdot}) + i\,(2ba + e + a^{\cdot})]\lambda^{-2}$$

which can be transformed into the following explicit expression:

$$W = \frac{1}{\sqrt{\sigma}}\left[C_1 \exp\left(\int_0^t \left(\frac{\chi}{2} - b - a\sigma \sin\frac{\varphi}{2}\right)dt + i\int_0^t a\left(1 + \sigma\cos\frac{\varphi}{2}\right)dt - \frac{i\varphi}{4}\right) + C_2 \exp\left(\int_0^t \left(\frac{\chi}{2} - b + a\sigma\sin\frac{\varphi}{2}\right)dt + i\int_0^t a\left(1 - \sigma\cos\frac{\varphi}{2}\right)dt - \frac{i\varphi}{4}\right)\right] - \frac{R_1 + iR_2}{c - ei}$$

$$\operatorname{tg}\varphi = (2ba + e + a^{\cdot})(a^2 - c - b^2 - b^{\cdot})^{-1}$$

For the solution to be bounded, we must have $\operatorname{Re}\tau^2 > 0$, $\operatorname{Im}\tau^2 \approx 0$, and this yields

$$a^2 - c > b^2 + b^{\cdot}$$

$$e < 0, \quad 2a\,(1 - \sigma)\left(1 - \frac{\chi}{2b}\right) < -\frac{e}{b} < 2a\,(1 + \sigma)\left(1 - \frac{\chi}{2b}\right)$$

$$e \geqslant 0, \quad -2a\,(\sigma + 1)\left(1 - \frac{\chi}{2b}\right) < \frac{e}{b} < 2a\,(\sigma - 1)\left(1 - \frac{\chi}{2b}\right)$$

In addition, the particular solution and C_1, C_2 must be sufficiently small in modulo, and this will only be the case when R_1 and R_2 in the interval $[0, T]$ and the initial perturbations are all sufficiently small. Then the deviations of the symmetry axis of the body from the tangent to the trajectory of its center of mass will also be small in the interval $[0, T]$.

Thus the conditions of boundedness of the solution of [7] represent the necessary conditions for the stability of the unperturbed motion (4), while the sufficient conditions are represented by the conditions of positive definiteness of the function V and negative definiteness of time derivative $V_{\mathbf{;}}$ by virtue of the simplified system of equations (3).

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